Solution Sheet 11

Exercise 1. Let X be a path-connected topological space and fix $x_0 \in X$. A loop $\gamma : S^1 \to X$ represents an element of $\pi_1(X, x_0)$ and determines a class in $H_1(X, \mathbb{Z})$. Define

$$\Phi_{x_0} : \pi_1(X, x_0) \to H_1(X, \mathbb{Z}), \quad \Phi_{x_0}([\gamma]) = [\gamma] \in H_1(X, \mathbb{Z}),$$

where on the right-hand side we write γ as shorthand for the induced 1-cycle $\gamma_\#(\sigma)$ for a fixed 1-cycle σ of S^1.

(1) Show that Φ_{x_0} is a well-defined group homomorphism.
(2) If $x_1 \in X$ and η is a path from x_0 to x_1, let $c_\eta([\gamma]) = [\eta^{-1} * \gamma * \eta] \in \pi_1(X, x_1)$. Show that the maps to homology are compatible with basepoint change, that is, $\Phi_{x_1} \circ c_\eta = \Phi_{x_0}$.
(3) Show that the map Φ_{x_0} induces an isomorphism

$$\pi_1(X, x_0)^{\text{ab}} \cong H_1(X, \mathbb{Z}),$$

where $\pi_1(X)^{\text{ab}} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ is the abelianization.

Solution 1. (1) Let $\gamma_0, \gamma_1 : (S^1, 1) \to (X, x_0)$ be loops based at x_0 representing the same element in $\pi(X, x_0)$. Thus they differ by a homotopy that keeps the basepoint fixed. That is, there is a homotopy

$$H : S^1 \times I \to X, \quad H(x, 0) = \gamma_0(x),\ H(x, 1) = \gamma_1(x)$$

and $H(1, t) = x_0$ for all t. In singular homology, a homotopy between cycles induces a chain homotopy between the associated chain maps. In particular, the homotopy H induces a 2-chain $K \in C_2(X)$ whose boundary is the difference of the loops:

$$\partial K = \gamma_{1\#}(\sigma) - \gamma_{0\#}(\sigma).$$

This implies that $[\gamma_0] = [\gamma_1] \in H_1(X, \mathbb{Z})$ and so Φ_{x_0} is well-defined.

Let $\gamma, \delta : S^1 \to X$ be loops at based at x_0. Let $S^1 \vee S^1$ be the wedge of two circles with inclusions $i_1, i_2 : S^1 \to S^1 \vee S^1$. Let $p : S^1 \to S^1 \vee S^1$ be the map that collapses a point of S^1 so that the first half of S^1 maps once around the first circle and the second half maps once around the second circle. Let $f : S^1 \vee S^1 \to X$ be the map with $f \circ i_1 = \gamma$ and $f \circ i_2 = \delta$. Then by construction $f \circ p : S^1 \to X$ is homotopic to the concatenation $\gamma * \delta$. Therefore

$$\Phi_{x_0}([\gamma * \delta]) = (\gamma * \delta)_\#([\sigma]) = (f \circ p)_\#([\sigma]) = f_\#\big(p_\#([\sigma])\big).$$

We have the identification $H_1(S^1 \vee S^1, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, where the two generators are $i_{1\#}([\sigma])$ and $i_{2\#}([\sigma])$. Since $p : S^1 \to S^1 \vee S^1$ goes once around each wedge circle we have

$$p_\#([\sigma]) = i_{1\#}([\sigma]) + i_{2\#}([\sigma]).$$

We then get

$$\begin{aligned}
\Phi_{x_0}([\gamma * \delta]) &= f_\#\, i_{1\#}([\sigma]) + f_\#\, i_{2\#}([\sigma]) \\
&= (f \circ i_1)_\#([\sigma]) + (f \circ i_2)_\#([\sigma]) \\
&= \gamma_\#([\sigma]) + \delta_\#([\sigma]) \\
&= \Phi_{x_0}([\gamma]) + \Phi_{x_0}([\delta]).
\end{aligned}$$

Thus Φ_{x_0} is a group homomorphism.

(2) Let η be a path from x_0 to x_1 and let γ be a loop based at x_0. For $t \in [0, 1]$ set $\eta_t(s) = \eta(ts)$. Then the loops

$$\ell_t := \eta_t^{-1} * \gamma * \eta_t$$

vary continuously with t and give a homotopy $H : S^1 \times [0,1] \to X$ with $H(z,t) = \ell_t(z)$. In particular, ℓ_0 is homotopic to γ and $\ell_1 = \eta^{-1} * \gamma * \eta$. Therefore $[\eta^{-1} * \gamma * \eta] = [\gamma]$ in $H_1(X, \mathbb{Z})$, i.e. $\Phi_{x_1}(c_\eta([\gamma])) = \Phi_{x_0}([\gamma])$.

(3) Since $H_1(X, \mathbb{Z})$ is an abelian group, the map

$$\Phi_{x_0} : \pi_1(X, x_0) \to H_1(X, \mathbb{Z})$$

must kill all commutators. This means that every commutator lies in the kernel of Φ_{x_0}, and so Φ_{x_0} factors uniquely through the abelianization:

$$\pi_1(X, x_0) \xrightarrow{q} \pi_1(X, x_0)^{\text{ab}} \xrightarrow{\overline{\Phi}_{x_0}} H_1(X, \mathbb{Z}).$$

We now show that $\overline{\Phi}_{x_0}$ is an isomorphism.

- *Surjectivity.* One can prove that every 1-cycle is homologous to a finite sum of loops. Hence every class in $H_1(X, \mathbb{Z})$ can be represented by such a sum.
- *Injectivity.* To prove $\overline{\Phi}$ is injective, one can construct a homomorphism

$$\Psi : H_1(X, \mathbb{Z}) \to \pi_1(X, x_0)^{\text{ab}}$$

such that $\Psi \circ \overline{\Phi} = \text{id}$. For each 1-simplex σ from a point p to a point q, choose fixed paths λ_p and λ_q from the basepoint x_0 to p and q, and define $\Psi_1(\sigma)$ as the abelianized homotopy class of the loop $\lambda_p * \sigma * \lambda_q^{-1}$. We extend this definition linearly to all 1-chains. One checks that the boundary of any 2-simplex gives a loop that is null-homotopic, hence Ψ_1 vanishes on boundaries and thus descends to a homomorphism $\Psi : H_1(X, \mathbb{Z}) \to \pi_1(X, x_0)^{\text{ab}}$. By construction, if γ is a loop based at x_0, then $\Psi([\gamma]) = q([\gamma])$. Consequently Ψ is a left inverse to $\overline{\Phi}$.

Exercise 2. (for credit, due on 7 December)
Consider the unit square $[0,1]^2$ with sides identified according to $aba^{-1}b^{-1}$. Let $\pi : [0,1]^2 \to T^2$ be the quotient map. Identify $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}\langle [a] \rangle \oplus \mathbb{Z}\langle [b] \rangle$, where $[a]$ is represented by the image of the bottom edge oriented left-to-right, and $[b]$ by the right edge oriented bottom-to-top. For each path α below, compute the homology class of the loop $\pi \circ \alpha$.

(1) *(1 point)* The diagonal from $(0,0)$ to $(1,1)$.
(2) *(1 point)* The diagonal from $(1,0)$ to $(0,1)$.
(3) *(1 point)* The loop obtained by traversing the three straight segments

$$(0,0) \to (1,0) \to (1,1) \to (0,0).$$

(4) *(2 points)* Fix coprime integers $p, q \in \mathbb{Z}_{\geq 1}$ and define

$$\alpha_{p,q}(t) = \big(pt \bmod 1, qt \bmod 1\big), \quad t \in [0,1].$$

Show that $\pi \circ \alpha_{p,q}$ is a loop with no self-intersections and compute its class in $H_1(T^2, \mathbb{Z})$.

Solution 2.

(1) We have $[\pi \circ \alpha] = [a] + [b] \in H_1(T^2, \mathbb{Z})$.
(2) We have $[\pi \circ \alpha] = [b] - [a] \in H_1(T^2, \mathbb{Z})$.
(3) We have $[\pi \circ \alpha] = [a] + [b] - ([a] + [b]) = 0 \in H_1(T^2, \mathbb{Z})$. Indeed, this loop bounds a 2-chain in the torus, hence is null-homologous.
(4) Since $p, q \in \mathbb{Z}$ we have

$$\alpha_{p,q}(0) = (0,0), \quad \alpha_{p,q}(1) = (p \bmod 1, q \bmod 1) = (0,0),$$

hence $\pi \circ \alpha_{p,q}$ is a loop. Suppose $(\pi \circ \alpha_{p,q})(t_1) = (\pi \circ \alpha_{p,q})(t_2)$ with $t_1, t_2 \in [0,1)$. Equality on $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ means

$$(pt_1, qt_1) - (pt_2, qt_2) \in \mathbb{Z}^2,$$

so letting $\Delta t = t_1 - t_2$ we have $p\Delta t \in \mathbb{Z}$ and $q\Delta t \in \mathbb{Z}$. Write $p\Delta t = m$ and $q\Delta t = n$ with $m, n \in \mathbb{Z}$. Then we have $qm = pn$. Because $\gcd(p, q) = 1$, it follows that $p \mid m$ and therefore we can write $m = kp$ for some $k \in \mathbb{Z}$. Thus $\Delta t = m/p = k \in \mathbb{Z}$. Since $\Delta t \in (-1, 1)$, this forces $\Delta t = 0$ and $t_1 = t_2$. Thus the loop has no self-intersections. Consider the lift $\widetilde{\alpha}(t) = (pt, qt)$ in $\mathbb{R}^2$. The endpoint difference is

$$\widetilde{\alpha}(1) - \widetilde{\alpha}(0) = (p, q),$$

and under the identification $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$ with $[a] \to (1, 0), [b] \to (0, 1)$, this yields

$$[\pi \circ \alpha_{p,q}] = p[a] + q[b] \in H_1(T^2, \mathbb{Z}).$$

Exercise 3. Show that if $\alpha \sim \alpha + \partial C$ in H_1, then $(\alpha + \partial C) \circ \beta = \alpha \circ \beta$ for every 1-cycle β.

Solution 3. Any boundary ∂C has zero intersection number with every cycle β: Every time β crosses the boundary ∂C, it must later cross back out. The intersection number at the entry crossing and exit crossing have opposite opposite signs. Because entry and exit crossings always come in pairs, the intersection number $\partial C \circ \beta$ is 0.

Exercise 4. Let X be a Riemann surface of genus g. Recall that X has a planar model with symbol $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$. We use the same symbols a_j, b_j to denote the corresponding homology classes in $H_1(X, \mathbb{Z})$. Denote by $\circ$ the intersection pairing on $H_1(X, \mathbb{Z})$.

(1) Show using the presentation

$$\pi_1(X) \cong \left\langle a_1, b_1, \dots, a_g, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

that

$$H_1(X, \mathbb{Z}) \cong \pi_1(X)^{\mathrm{ab}} \cong \mathbb{Z}^{2g}.$$

(2) Show that

$$a_i \circ b_j = \delta_{ij}, \quad a_i \circ a_j = 0 = b_i \circ b_j \text{ for all } i, j.$$

(3) A canonical basis of $H_1(X, \mathbb{Z})$ is a basis $(a_1, \dots, a_g, b_1, \dots, b_g)$ satisfying the identities in (2). Let $a'_1, \dots, a'_g, b'_1, \dots, b'_g$ be another canonical basis of $H_1(X, \mathbb{Z})$. Show that there is a matrix $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that

$$\begin{pmatrix} a'_1 \\ \vdots \\ a'_g \\ b'_1 \\ \vdots \\ b'_g \end{pmatrix} = M \begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b_1 \\ \vdots \\ b_g \end{pmatrix}.$$

Solution 4.

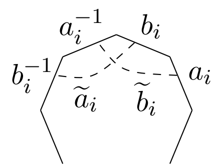
(1) After abelianization, each commutator $[a_i, b_i]$ in the presentation of $\pi_1(X)$ becomes 1, so the single relation $\prod_{i=1}^{g} [a_i, b_i] = 1$ imposes no constraint. Therefore

$$\pi_1(X)^{\mathrm{ab}} \cong \langle a_1, b_1, \dots, a_g, b_g : \text{ all commute } \rangle \cong \mathbb{Z}^{2g}.$$

From exercise 1 we deduce that $[a_1], [b_1], \dots, [a_g], [b_g]$ form a basis of $H_1(X, \mathbb{Z})$.

(2) All vertices of the $4g$-gon are identified to one point of X, so the edge-loops a_i, b_i meet there and are not transverse. For each i, choose instead an embedded arc in the polygon joining the two sides labeled b_i and b_i^{-1}. After the identification $b_i \sim b_i^{-1}$ the endpoints are identified, so it becomes a closed loop $\widetilde{a}_i \subset X$. Similarly, choose an embedded arc joining a_i to a_i^{-1}, giving a closed loop $\widetilde{b}_i \subset X$. By construction $\widetilde{a}_i$

is homotopic to a_i and similarly $\widetilde{b}_i$ is homotopic to b_i. Consequently $[\widetilde{a}_i] = [a_i]$ and $[\widetilde{b}_i] = [b_i]$ in $H_1(X, \mathbb{Z})$.



We can choose these arcs so that $\widetilde{a}_i$ and $\widetilde{b}_i$ intersect transversely in exactly one point, and so that for $i \neq j$ all curves $\widetilde{a}_i, \widetilde{b}_i$ are pairwise disjoint. This is possible since different blocks $a_i b_i a_i^{-1} b_i^{-1}$ are disjoint in the polygon. We choose the orientation of X so that the unique intersection of $\widetilde{a}_i$ and $\widetilde{b}_i$ is positive. This gives

$$\widetilde{a}_i \circ \widetilde{b}_j = \delta_{ij}, \qquad \widetilde{a}_i \circ \widetilde{a}_j = 0, \qquad \widetilde{b}_i \circ \widetilde{b}_j = 0.$$

Since the intersection pairing depends only on homology classes, the same equalities hold for a_i, b_j.

(3) We may express the cycles a_i', b_i' in terms of the basis a_i, b_i. In matrix form this means that

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M \begin{pmatrix} a \\ b \end{pmatrix}$$

for a unique matrix $M \in \mathrm{GL}_{2g}(\mathbb{Z})$. Uniqueness and integrality follow since both $(a_1, \dots, a_g, b_1, \dots, b_g)$ and $(a_1', \dots, a_g', b_1', \dots, b_g')$ are $\mathbb{Z}$-bases of $H_1(X, \mathbb{Z})$. Let $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ denote the intersection matrix of a canonical basis, i.e. so that

$$J_{kl} = e_k \circ e_l \text{ for } (e_1, \dots, e_{2g}) = (a_1, \dots, a_g, b_1, \dots, b_g).$$

Since (a_i', b_i') is also canonical, its intersection matrix is the same:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \circ \begin{pmatrix} a' & b' \end{pmatrix} = J, \qquad \begin{pmatrix} a \\ b \end{pmatrix} \circ \begin{pmatrix} a & b \end{pmatrix} = J.$$

We then get

$$J = \begin{pmatrix} a' \\ b' \end{pmatrix} \circ \begin{pmatrix} a' & b' \end{pmatrix} = \left(M \begin{pmatrix} a \\ b \end{pmatrix} \right) \circ \left(\begin{pmatrix} a & b \end{pmatrix} M^T \right)$$
$$= M \left(\begin{pmatrix} a \\ b \end{pmatrix} \circ \begin{pmatrix} a & b \end{pmatrix} \right) M^T = M J M^T.$$

Hence $MJM^T = J$, so $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$.